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# Are Indexed Bonds Really Inflation Proof? A Model of Real and Nominal Term Structures when Money has Real Effects

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**Abstract.** This paper studies the general behavior of the nominal and real term structures of interest rates in a general equilibrium framework. A central bank is introduced in the model as an agent facing a tradeoff between inflation and output and choosing a monetary policy variable. Prices and output are jointly determined in our model endogenously. Two multi-factor nominal and real term structure models are given as examples to illustrate the general model. In our economies, inflation indexed bonds are not completely inflation proof, but are still subject to the influence of inflation uncertainties. The models offer us an empirical framework that can be studied with indexed bond data and nominal bond data together in a single estimation.

**Key words:** inflation, inflation-indexed bond, term structure models, asset pricing

**JEL Classification:** G12, E31

## Introduction

The United States Treasury Department started issuing "inflation-indexed" bonds in January of 1997. These bonds are designed to provide a guaranteed real rate of return that is not subject to inflation fluctuation. In this case, inflation is no longer a direct factor that affects the return on these bonds. However, indexed bonds may still not be inflation proof, because inflation may have a real effect on the return of indexed bonds indirectly by affecting real production. Because the term structure of real interest rates is as crucially important to the pricing of indexed bonds as the term structure of nominal interest rates is to nominal bonds, understanding relations among expected inflation, actual inflation, real interest rates, and

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nominal interest rates is a first step toward any further study of indexed bonds. This may also yield insights into other macroeconomic phenomena.

A multi-factor nominal and real term structure model is presented in this paper in the general equilibrium framework offered by Cox, Ingersoll and Ross (1985a, CIR henceforth), with emphasis on the role played by the monetary authority and on the implications of the relations among inflation, employment, output, and the term structure. There have been many attempts to study the relation between the real and nominal term structure of interest rates in a modern framework. CIR (1985b) first formulated a model in which an exogenous price level process is given to be independent of other economic fundamentals. Thus money is neutral in their economy, and consequently, real interest rates are not affected by inflation at all. Richard (1978) presented a similar model in an arbitrage pricing framework. These works have laid the foundation for other contributions. Sun (1992) argued in his partial equilibrium model that money is not neutral. In his model, an exogenous price level process is correlated with the real economy. He also found a positive correlation between the consumption growth rate and the price level growth rate. Pennacchi (1991) modeled the output process and price process as a state-space system that is affected by two common factors. He found a negative correlation between instantaneous real interest rates and expected inflation. Brown and Schaefer (1994) studied British inflation index-linked bond data and found a different behavior of real interest rates. Specifically, they find that long-term real interest rates are much more stable than their nominal counterparts. In other words the term structure of real interest rates is more volatile than the nominal term structure. There are also several works in related areas. Bakshi and Chen (1996) put real money balances into the utility function to justify the movement of nominal interest rates. In their model, the price process is derived endogenously for the first time; but, the money supply process is given exogenously. Because real money balances facilitate transactions, the money supply can alter the utility levels of consumers without affecting real output. Thus, the real and nominal term structure of interest rates are independent of each other. Their model shed some lights on understanding the economic forces driving the real and nominal interest rates and inflation. It also built a bridge between powerful tools in monetary economics and the financial economics theory.

All of the above models, except Bakshi and Chen (1996), took the price process as exogenous. The structures were directly imposed by these authors in order to maintain mathematical tractability and serve the intentions of the models. The difficulty of these models is that in continuous-time, our ability to make variables exogenous is very limited. The Bakshi and Chen (1996) model is attractive in terms of price endogeneity and the economics behind the model, but the result that nominal and real interest rates are independent is less than satisfactory. Other models that allow more complicated structures, such as Pennacchi (1991), however, lack economic explanations of what is behind the structures of models.

Our model attempts to solve the problems of the existing literature and still maintains the tractable properties of other models. First of all, price is endogenous in our model. It is jointly determined with output in equilibrium. This is achieved by adding a central bank as an agent in the economy. In monetary economics, there is a large body of literature devoted to the study of the central bank's behavior which assumes multiple objectives in the bank's utility function. We assume that the central bank wants both a low inflation rate and high employment growth. It chooses its monetary policy variable by minimizing its intertemporal loss function involving its inflation and employment objectives. The choice of the monetary policy, represented by one of the state variables in our model, is determined by the market condition when the central bank makes monetary policy decision. We achieve endogeneity by using stochastic processes, but leave undetermined coefficient in the drift and diffusion terms of the processes. We claim that price and output are endogenous in the sense that we have to solve for the undetermined coefficient in the processes that govern the price and output.

The second main contribution of our model is that despite the economic intuition behind the model, the results of our model are rather general. Real and nominal interest rates and expected inflation are all correlated with each other. Instead of being imposed by authors as in previous models, these results are derived from the real and nominal sectors of the economy. The state variables in the term structure literature are usually unspecified. In our model, all the state variables have economic meanings. This way, we can significantly improve our understanding of the term structure in the context of the whole economy.

The third contribution comes from the mathematical tractability of the model. Because we have closed-form solutions for the prices of a wide range of real and nominal bonds and their derivatives, this model can be readily tested. Ten years of British indexed and nominal bond data are a natural starting point. This model also allows us to utilize other macro data, such as inflation data and other monetary policy variables with bond data in estimating parameters and testing hypotheses. These features all stem from our emphasis on economic variables in a term structure model.

This paper is organized as follows: In Section 2, we show the general model and give the general solutions determining the central bank's policy, real and nominal interest rates and real and nominal asset prices. Section 3 specializes the model in a three-factor Vasicek framework (Vasicek, 1977). In this mathematically tractable model, we can let the structure of the economy be very flexible, yet we can still obtain closed-form solutions. The role played by each economic variable is explored in great detail. In Section 4, we show a more realistic model that satisfies some natural constraints, i.e., the nominal interest rate cannot fall below zero, but the real interest rate and expected inflation can possibly be negative. Unlike other models, the role of the speculative demand of money is implied by this model without sacrificing the possibility of deflation and negative real return. Money as an asset does not allow a negative interest rate on nominal bonds. We also solve for bond prices in a mixed square-root and Vasicek setup. Section 5 discusses the potential empirical study of our model. Conclusions are reached and further research directions are suggested in Section 6.

## The model

In this section, we first define the economy, including the consumer preferences, the production, and the central bank's objectives and action. The central bank will decide its monetary policy according to the market condition in the beginning. Then we solve the general equilibrium model for the nominal and real interest rates and other asset prices. The parametric forms of this model are given in the next two sections.

*The economy*

The economy consists of identical producers, identical consumers, and a benevolent central bank. There is only one good in the economy. This single good can either be consumed or be used as capital input in the production technology combined with labor input to provide the same good in the next instant. There is also money in the economy to facilitate trades. Trades of goods and financial assets can occur and be denominated either in the consumption good or in money. The central bank controls monetary policy. Monetary policy will affect both real output and nominal prices. There is no storage technology in this economy, so the only means to carry wealth intertemporally is to invest in real production for the society as a whole. But individuals can invest in real production, real assets, and nominal assets including money. There is no asymmetric information problem in this economy. Both the consumers and the central bank are rational. However, there are nominal unmodeled rigidities like downward wage rigidity[1] in the economy so that monetary policy can have real effects on the labor markets, and therefore real output.

Consumers in this economy have identical time-additive constant relative risk aversion utility functions with relative risk aversion parameters equal to 1 (CRRA-1). Their intertemporal utility function is given by

$$E_t\left[\int_t^{+\infty} e^{-\rho s}\ln(C_s)\,ds\right], \tag{1}$$

where $\rho$ is the constant discount factor.[2] With this special utility function, a representative consumer will always consume a constant fraction, $g'$, of his total wealth, as shown by Merton (1971).

We let the stochastic productivity process be exogenously given by

$$dA_t = \mu_{a,t}(\mathbf{x}_t)A_t\,dt + \sigma_{a,t}(\mathbf{x}_t)A_t\,dB_{a,t}, \tag{2}$$

where $\mu_{a,t}$ and $\sigma_{a,t}$ are expected mean and standard deviation of the productivity growth at time $t$. $\mathbf{x}_t$ is a vector of exogenous state variables. $\mu_a$ and $\sigma_a$ are functions of these state variables. $B_{a,t}$ is a standard Wiener process. The employment level follows

$$dN_t = \mu_{n,t}(\mathbf{x}_t, q)N_t\,dt + \sigma_{n,t}(\mathbf{x}_t, q)N_t\,dB_{n,t}, \tag{3}$$

where $\mu_{n,t}$ and $\sigma_{n,t}$ are continuous processes denoting the expected mean and standard deviation of the labor growth rate at time $t$. They will be affected by the state variables and also by $q$, the monetary policy variable that affects both the inflation and the employment. We will define this variable when we start to talk about inflation. $B_{n,t}$ is another standard Wiener process correlated with $B_{a,t}$. The correlation coefficient is $\rho_{an}$. When $q$ is set to be a number such that price is stabilized, the employment level will have natural employment growth rates. The natural employment growth rates are not necessarily the socially desirable growth rates because of the nominal rigidities we have assumed. The individuals in this economy are endowed with one production technology that requires physical inputs $Y_t$ and labor inputs $N_t$. Together with the exogenous productivity $A_t$, it produces the same good as the output $Y_{t+dt}$. The stochastic process that determines the return to physical investment in this technology is determined by the productivity growth process and employment growth process[3]

$$Y_t = A_t N_t. \tag{4}$$

Applying Ito's lemma to Eq. (4), the dynamic of this production function becomes

$$dY_t = Y_t \frac{dA_t}{A_t} + Y_t \frac{dN_t}{N_t} + \rho_{an}\sigma_{a,t}\sigma_{n,t} Y_t\, dt. \tag{5}$$

Individuals will decide how much, $C_t$, they want to consume and how much to reinvest. Since physical investment coincides with final output, when we have continuous production function with consumers who have CRRA-l utility functions that generate consumption as a constant fraction of total wealth, physical investment will be continuous as well.

Assume that the nominal price of the consumption good is given by the process

$$dP_t = \mu_{p,t}(\mathbf{x}_t, q) P_t\, dt + \sigma_{p,t}(\mathbf{x}_t, q) P_t dB_{p,t}, \tag{6}$$

where $B_{p,t}$ is another standard Wiener process having correlation coefficients $\rho_{ap}$ and $\rho_{np}$ with the Wiener processes governing the productivity and employment uncertainties. $\mu_{p,t}$ is the expected price growth rate or expected inflation. $\sigma_{p,t}$ is the unexpected volatility of the price growth rates at time $t$. $\mu_{p,t}$ and $\sigma_{p,t}$ are functions of the state variables and the monetary policy variable, $q$. Both the expected and unexpected change of prices are affected by the central bank's monetary policy through different possible channels such as open market operations, reserve requirement, and discount window lending procedures. Then in an economy with perfect foresight, expected inflation is the rationally expected part of the actual inflation. If we let $\omega_t$ represent actual inflation rate, then it follows

$$d\omega_t = \frac{dP}{P} = \mu_{p,t}(\mathbf{x}_t, q)\, dt + \sigma_{p,t}(\mathbf{x}_t, q)\, dB_{p,t}, \tag{7}$$

In an economy with rational agents, expected inflation is the expected price growth rate, $\mu_{p,t}$ in Eq. (7).

Inflation plays different roles in the whole economy. It is argued that inflation can "grease the wheels of the labor market". Inflation can lower the real wage without lowering the nominal wage such that even with downward nominal wage rigidity, employers do not have to fire workers in order to lower their costs. On the other hand, other than the downward wage rigidity effect that favors a higher inflation to keep the real economy unhurt, there might be other effects such as distributional effects that make people dislike inflation even if the inflation raises the level of real output. We assume that consumers take inflation as given and do not put any measure of inflation into their utility functions. We also introduce the central bank into our economy as the agent to make monetary policy decision.

Let us now model the central bank's behavior. The central bank cares about both price stability and employment or output growth. We specify two preference variables to represent both objectives that will enter into the central bank's utility function. Here, $\pi_t$ denotes inflation preference and $l_t$ denotes employment growth preference. Because both the growth rates and volatilities of the price process and employment process are stochastic, we let

$$\pi_t = f_1(\mu_{p,t}, \sigma_{p,t}), \tag{8}$$

$$l_t = f_2(\mu_{n,t}, \sigma_{n,t}). \tag{9}$$

Both the expected growth rates and the volatilities affect these functions, because not only the average growth rates but also the volatility matters. Also we let $\pi_t^*$ and $l_t^*$ be the targets of these two variables. Conventionally, $\pi_t^*$ denotes a price stability state and $l_t^*$ denotes an above natural employment growth trend. Finally, the loss function of the central bank is given by

$$E_t\left\{\int_t^{+\infty} e^{-\delta s}(\omega_1(\pi_s - \pi_s^*)^2 + \omega_2(l_s - l_s^*)^2)\,ds\right\} \tag{10}$$

where $\omega_1$ and $\omega_2$ are the weights on losses from inflation and employment being different from their targets, and $\delta$ is the intertemporal discount factor. This loss function is similar to the most conventional objective functional forms in the central banking literature which put levels of inflation and output objective in the loss function instead of preferences involving the drifts and volatilities of inflation and employment processes.[4]

In order to simplify the problem, we do not index the monetary policy variable by time. Instead of deciding on monetary policy continuously, the central bank has to choose the magnitude of the policy variable at one time, and follow this policy thereafter. We are not dealing with time consistency problem here. Once the central bank has chosen a policy, he has no incentive to deviate from its rule. The central bank's problem is to choose $q$ at $t_0$ in order to minimize (10).

The current monetary policy variable is given by the solution to the following equation:

$$\int_{t_0}^{+\infty} e^{-\delta s}\left(\omega_1 \frac{dE_t[(\pi_s(\mathbf{x}_s, q) - \pi_s^*)^2]}{dq} + \omega_2 \frac{dE_t[(l_s(\mathbf{x}_s, q) - l_s^*)^2]}{dq}\right) ds = 0. \tag{11}$$

**Proof:** See Appendix. □

If the central bank has an anti-inflation and above natural employment objective, Eq. (11) will deliver a solution such that the marginal intertemporal loss from an inflation rate above the bank's inflation target is equal to the marginal intertemporal gain from the above natural employment growth rate.

*The real interest rate and real asset prices*

The consumers are going to solve for their consumption and investment portfolio problem taking monetary policy as given. In this subsection, all assets are denominated in real consumption goods. In this economy, there is only one production technology available, and there is no storage technology. People can trade different real contingent claims including real free bonds.[5] All the real assets are in zero net supply except the real investment in the only production. However, these assumptions will not prevent us from calculating real interest rates and real asset prices. The short-term real interest rate is the instantaneous rate of return of real consumption good on one unit of riskless real investment. Let $W$ be the real wealth of a representative agent in terms of real consumption goods. His problem would be to maximize (1) subject to the budget constraint given by:

$$dW_t = W_t \frac{dY_t}{Y_t} - C_t\, dt. \tag{12}$$

The consumption process is maintained at a constant, $\rho$, proportion of the net wealth with level $\rho W$. Thus, the budget constraint (12) becomes

$$dW_t = (\mu_{a,t} + \mu_{n,t} + \rho_{an}\sigma_{a,n}\sigma_{n,t} - \rho)\, W_t\, dt + \sigma_{a,t} W_t dB_{a,t} + \sigma_{n,t} W_t dB_{n,t}. \tag{13}$$

Due to the tractability of the time-separable log-utility function, Merton (1971) shows that in solving the consumers' optimization problem, the indirect utility function can be written as

$$J(W_t, \mathbf{x}_t, q, t) = \rho^{-1} e^{-\rho t} \ln(W_t) + g(\mathbf{x}_t, q, t). \tag{14}$$

$J$ is a function of real wealth $W$ and other state variables. $J$ is separable such that $W$ is not an argument of $g$. Following Merton (1971), CIR (1985a) Theorem 1, or following Breeden (1986) by relating interest rates to marginal utility, we can solve for the short-term real interest rate process and the pricing formula for contingent claims. The real short-term interest rate is given by:

$$r_t = \mu_{a,t} + \mu_{n,t} - \rho_{an}\sigma_{a,t}\sigma_{n,t} - \sigma_{a,t}^2 - \sigma_{n,t}^2. \tag{15}$$

We can see that the real interest rate is equal to the expected growth rate of wealth minus the variance of the unexpected rate of change of wealth growth. In terms of factor inputs, real interest rates are equal to the expected rates of productivity growth times the relative fraction of productivity versus the physical capital, plus the expected growth rate of employment, minus the covariance of the unexpected change of productivity and employment scaled by their factor loadings, minus the variance of the unexpected rates of productivity growth times a constant, minus the variance of the unexpected changes of employment growth. The real interest rates can also be understood as minus expected rate of change in the marginal utility of wealth as first proposed by Breeden (1986). Monetary policy affects real interest rates by affecting the expected and unexpected rates of change of employment. We have not quantified the employment growth process, so we cannot study the effect of monetary policy in great detail. By now, we know at least that monetary policy is not neutral as long as its effect on employment are not negligible in this economy.

We can characterize the price of any contingent claim either by specifying the partial differential equation (PDE), or by finding the equivalent martingale measure. Together with proper initial and boundary conditions, we can solve for the price of any asset analytically or numerically. The PDE for any contingent claim to follow under the conditions of this economy is given by CIR (1985a). We will give a simple general equivalent martingale formula that is equivalent to the PDE approach. The task for us is to figure out the equivalent martingale for this formula.[6] If a contingent claim $F(\cdot)$ does not depend on the real wealth level, then it can be expressed as

$$F_t = E_t^Q\left\{\exp\left(\int_t^T -r_s\,ds\right)F_T + \int_t^T \exp\left(\int_t^s -r_z\,dz\right)d\delta_s\right\}, \tag{16}$$

where $\delta_s$ is the time-s dividend of asset $F$. Q is the equivalent martingale that can be derived both from general equilibrium models and arbitrage models.[7] Specifically, this Q will be adjusted by the covariance between the unexpected components of the exogenous state variables and the unexpected growth of the real wealth.

*Nominal interest rates and nominal asset prices*

Now we turn our attention to the nominal economy. In this subsection, the representative consumer has all his wealth denominated in nominal money, and can either spend his money on the consumption good or invest his money in nominal financial assets. The nominal short interest rate can be understood as the return to an instantaneous riskless nominal investment, the nominal bond that guarantees to pay back one unit of money. In equilibrium, all the nominal assets including the nominal bond are in zero supply except the stock that denotes the only production technology. Let $\varepsilon_t$ denote the time-$t$ representative consumer's nominal expenses on the consumption good. Given the price level $P_t$, his objective function can be written as

$$E_t\left[\int_t^{+\infty} e^{-\rho s}\ln\left(\frac{\varepsilon_s}{P_s}\right)ds\right]. \tag{17}$$

If we measure production by nominal wealth as well, the nominal output $Q_t$ at time $t$ can be written as

$$Q_t = Y_t P_t. \tag{18}$$

Let $Z_t$ be his nominal wealth level at time $t$. Then, $Z_t$ is given by

$$Z_t = W_t P_t. \tag{19}$$

Thus, the intertemporal budget constraint becomes

$$dZ_t = Z_t\frac{dQ_t}{Q_t} - \varepsilon_t\,dt. \tag{20}$$

Similar to the real economy case we just discussed, the indirect utility function can be written as

$$L(Z_t, \mathbf{x}_t, q, t) = \rho^{-1} e^{-\rho t} \ln(Z_t) + g'(\mathbf{x}_t, q, t). \tag{21}$$

$L$ is separable such that $Z$ is not an argument of $g'$. In fact, the consumer will spend a constant fraction, $\rho$, of his nominal wealth on consumption good at each instant at price $P_t$. The budget constraint becomes

$$\begin{aligned} dZ_t = {} & (\mu_{a,t} + \mu_{n,t} + \mu_{p,t} + \rho_{an}\sigma_{a,t}\sigma_{n,t} + \rho_{ap}\sigma_{a,t}\sigma_{p,t} + \rho_{np}\sigma_{n,t}\sigma_{p,t} - \rho) Z_t\, dt \\ & + \sigma_{a,t} Z_t dB_{a,t} + \sigma_{n,t} Z_t dB_{n,t} + \sigma_{p,t} Z_t dB_{p,t}. \end{aligned} \tag{22}$$

Similar to the real economy case, we can solve for the nominal interest rate process as

$$\begin{aligned} \iota_t = {} & \mu_{a,t} + \mu_{n,t} + \mu_{p,t} - \rho_{an}\sigma_{a,t}\sigma_{n,t} - \rho_{ap}\sigma_{a,t}\sigma_{p,t} - \rho_{np}\sigma_{n,t}\sigma_{p,t} \\ & - \sigma_{a,t}^2 - \sigma_{n,t}^2 - \sigma_{p,t}^2. \end{aligned} \tag{23}$$

The nominal interest rate is equal to the real interest rate plus the expected inflation rate minus the variance of the unexpected inflation rate, minus the covariance between the unexpected price growth rate and the unexpected productivity growth rate, minus the covariance between the unexpected price growth rate and the unexpected employment growth rate. The nominal interest rate can also be seen as the expected nominal wealth growth rate minus the variance of the unexpected wealth growth rate. Notice that monetary policy enters into the nominal interest rate through two channels, the employment growth rate and inflation.

Let us study real return on an asset that pays the nominal interest rate at any moment of time. First we define a money market account $M_t$ as such an asset. Its dynamics is given by:

$$dM_t = \iota_t M_t\, dt. \tag{24}$$

The real return $\rho^W$ is given by

$$R_t = \frac{M_t}{P}.$$

Applying Ito's lemma, we obtain

$$\begin{aligned} dR_t &= \left(\iota_t - \mu_{p,t} + \sigma_{p,t}^2\right) R_t\, dt + \sigma_{p,t} R_t\, dB_{p,t} \\ &= (\iota_t - \rho_{ap}\sigma_{a,t}\sigma_{p,t} - \rho_{np}\sigma_{n,t}\sigma_{p,t}) R_t\, dt + \sigma_{p,t} R_t\, dB_{p,t}. \end{aligned} \tag{25}$$

From this result, we can see that investors wound not generally receive a premium on the unexpected inflation uncertainty when they hold money market account. The only risk premium they receive for holding this asset is from the correlation between the unexpected inflation and the unexpected uncertainty in real output. If unexpected inflation is uncorrelated with real output, from the CAPM point of view, this uncertainty can always be diversified, thus investors need not to be compensated for bearing this risk.

Any financial asset $G(\cdot)$ that carries off its payoffs in terms of nominal money can be priced by the following equation

$$G_t = E_t^{Q'}\left\{\exp\left(\int_t^T -\iota_s ds\right)G_T + \int_t^T \exp\left(\int_t^s -\iota_z dz\right)d\delta'_s\right\}, \tag{26}$$

where $\delta'_s$ is the time-s dividend and $G_T$ is the time $T$ price of asset $G$ in monetary terms. $Q'$ is the equivalent martingale adjusted by the covariance of the unexpected growth of nominal wealth and unexpected component of other state variables.[8]

The monetary policy variable affects both real and nominal asset prices by affecting the real and nominal interest rates and by changing the risk adjustment of equivalent martingale measures. Now that we have been equipped with all the general solutions in real and nominal economies, we are ready to illustrate the interplay between the real and nominal interest rates, asset prices, inflation, and output in greater detail by examples.

## A three-factor Vasicek model

We will present a parametric form to illustrate the general model we introduced in the last section. Under the assumptions of this section, if the central bank chooses the monetary policy variables to cause a higher money growth rate, a higher expected inflation and a higher expected labor growth rate will be induced proportionally. The case of choosing a zero monetary policy variable corresponds to zero expected inflation and natural employment growth rate. Because the central bank has zero inflation and above natural employment growth objectives, the equilibrium price and employment will generally follow modest inflation and above natural employment growth paths. However, under very special circumstances, there might be deviations. Thus, the equilibrium real and nominal interest rates processes are affected by the central bank's choice of the policy variable. These assumptions are supported by the argument that inflation (or at least modest inflation) greases the wheels of labor markets. Thus employment will grow at a faster pace than the natural employment growth rate. Here, the natural employment growth rate is not socially optimal. Next, we will formally introduce the economy.

### *The economy*

The economy is characterized by the productivity, employment, and price processes as given by

$$dA_t = x_{1,t}A_t\,dt + \sigma_1 A_t\,dB_{a,t}, \tag{27}$$

$$dN_t = (x_{2,t} + q\eta x_{3,t})N_t\,dt + \sigma_2 N_t\,dB_{n,t}, \quad \text{and} \tag{28}$$

$$dP_t = q(1-\eta)x_{3,t}P_t\,dt + \sigma_3 P_t\,dB_{p,t}, \tag{29}$$

where $\sigma_i s$ are positive constants and the state variables $x_i s$ are given by

$$dx_{1,t} = k_1(\theta_1 - x_{1,t})\,dt + \nu_1\,dB_{1,t}, \quad (30)$$
$$dx_{2,t} = k_2(\theta_2 - x_{2,t})\,dt + \nu_2\,dB_{2,t}, \quad \text{and} \quad (31)$$
$$dx_{3,t} = k_3(\theta_3 - x_{3,t})\,dt + \nu_3\,dB_{3,t}. \quad (32)$$

where $\eta$, $k_i s$, $\theta_i s$ and $\nu_i s$ are positive constants. $B'_{\cdot,t}s$ are standard Wiener processes. Any pair of Wiener processes, $B_{i,t}$ and $B_{j,t}$, are allowed to be correlated. Their correlation coefficient is $\rho_{ij}$ and $x_i$ is a mean-reverting Brownian motion with unconditional long-run mean $\theta_i$, decaying parameter $k_i$, and diffusion parameter $\nu_i$. This process is also known as Ornstein-Uhlenbeck process in the stochastics literature and Vasicek process in the finance literature. Agents in the economy can observe $x_{it}$ at time $t$, but not $B_{a,t}$, $B_{n,t}$, and $B_{p,t}$. Technology has a mean reverting expected growth rate and a constant unexpected growth rate. The expected growth rate of employment is equal to the sum of the expected natural employment growth rate and a factor reflecting the real effects of monetary policy. The unexpected growth of employment is constant. The expected inflation is equal to the total nominal effect of the monetary policy minus the effect on employment. The unexpected inflation rate is constant as well. In this setup, the total monetary policy effect on the economy is divided into constant shares to affect expected employment growth rate and expected inflation, say, $\eta$ of the total effect on expected employment growth rate, and $1-\eta$ of the total effect on expected inflation rate. If the central bank chooses a larger $q$, the effect would be that the expected inflation would have a higher unconditional mean, a higher volatility, and the same frequency, or decaying speed. Precisely, the expected inflation is given by

$$\omega_t^e = q(1-\eta)\,x_{3,t}, \quad (33)$$

and

$$d\omega_t^e = k_3(q(1-\eta)\theta_3 - \omega_t)\,dt + q(1-\eta)\nu_3\,dB_{3,t}. \quad (34)$$

At the same time, the expected employment growth goes up by $q\eta x_{3,t}$. The long-run mean, volatility, and decaying speed of expected employment growth are all affected by the monetary policy. However, no matter what value the central bank chooses the policy variable $q$ to be, the unexpected rates of changes of the economic variables: productivity, employment, and price will not change.

*Monetary policy*

Let us now consider what will enter into the central bank's objective function. Aside from the expected inflation, it should also care about the unexpected price growth. However, since the expected change part is not at its disposal, we can only put the expected inflation into its objective. Similarly, the labor growth variable is represented by the expected labor growth rate. So we have the conventional definitions

$$\pi_t = \mu_{p,t} = q(x_{3,t} - \mu_t), \tag{35}$$
$$l_t = \mu_{n,t} = x_{2,t} + q\mu_t. \tag{36}$$

We also assume that the central bank has a zero target for expected inflation

$$\pi_t^* = 0. \tag{37}$$

Similar to the assumptions often used in monetary economics literature that the central bank wants to keep employment above the natural level, we assume that the benevolent central bank has an above natural employment growth objective,

$$l_t^* = x_{2,t} + b, \tag{38}$$

where $b$ is a constant that represents by how much it is ideal for the central bank to set the employment growth rate above the natural rate. This assumption is justified by the argument that natural employment growth rate is suboptimal due to the nominal rigidities.

Here, the central bank has to decide on his monetary policy rule at time $t_0$, which is fully characterized by the parameter $q$ based on minimizing his loss function (10). He has to choose a policy to serve both of his objectives, maintaining high output growth and low inflation. Based on the assumptions above, we can solve for the policy variable $q$.

**Proposition 1.** *The central bank will choose the policy variable $q^*$ at time $t_0$ as*

$$q^* = \frac{\omega_2 \eta b \mathbf{A}_3(t_0)}{(\omega_1(1-\eta)^2 + \omega_2\eta^2)\mathbf{B}_3(t_0)}, \tag{39}$$

*where*

$$\mathbf{A}_i(s) = \frac{x_{i,s} - \theta_i}{k_i + \delta} + \frac{\theta_i}{\delta}, \quad \textit{and}$$
$$\mathbf{B}_i(s) = \frac{(x_{i,s} - \theta_i)^2 - \frac{v_i}{2k_i}}{2k_i + \delta} + \frac{2\theta_i(x_{i,s} - \theta_i)}{k_i + \delta} + \frac{\frac{v_i}{2k_i} + \theta_i^2}{\delta}.$$

**Proof:** See Appendix. □

In this example, $q^*$ is positive when $x_{3,t_0} > -k_3\theta_3/\delta$. That is, the central bank will choose a long-run inflation unless the current realization of $x_{3,t_0}$ is too small. The reason is that we allow short-run deflation to have a negative effect on employment. Thus, when the state variable turns out to be a big negative number, the central bank will choose a negative policy value such that it can gain by keeping a short-run inflation, but in the long-run, the unconditional distribution of expected inflation will have a strong deflationary effect.

*The interest rates and asset prices*

We can solve for interest rates and asset prices in this economy. The interest rates are given in the following proposition.

**Proposition 2.** *The real short interest rate is given by*

$$r_t = x_{1,t} + x_{2,t} + q\eta x_{3,t} - \rho_{an}\sigma_1\sigma_2 - \sigma_1^2 - \sigma_2^2. \tag{40}$$

*The nominal short interest rate is given by*

$$r_t = x_{1,t} + x_{2,t} + qx_{3,t} - \rho_{an}\sigma_1\sigma_2 - \rho_{ap}\sigma_1\sigma_3 - \rho_{np}\sigma_2\sigma_3 - \sigma_1^2 - \sigma_2^2 - \sigma_3^2. \tag{41}$$

**Proof:** Applying Eq. (15) directly to (27) through (32). □

In this example, if the nominal economy is expected to grow $qx_{3,t}$ at time $t$, inflation is expected to grow $(1-\eta)qx_{3,t}$ and employment, thus output, is expected to grow $\eta qx_{3,t}$ above their natural growth rates. As shown in the interest rates, the central bank's influence over the economy is reflected by the $qx_{3,t}$ component in the nominal interest rate, and by the $\eta qx_{3,t}$ component in the real interest rate. Similar to the expected inflation rate, monetary policy $q$ can change the long-run mean and volatility of both real and nominal interest rates, but not the mean reverting coefficients of the components of the interest rates. In other words, a higher value of monetary policy will give higher means and volatilities to the interest rates, but not the periods of the cyclical behavior of the state variables. However, the mixed cyclical behavior will be different due to the different factor loadings of the different state variables. The nominal interest rate is equal to the sum of real interest rate and the expected inflation minus the variance of the unexpected rate of change of the price process minus the covariances between the unexpected change of price growth and unexpected change of productivity and employment growth rates. This is because people discount the new unexpected shocks introduced by the price uncertainty. From Eq. (25), we can see that the expected real return on the money market account is $r_t - \rho_{ap}\sigma_1\sigma_3 - \rho_{np}\sigma_2\sigma_3$.

With the parametric setup in this section, the equivalent Martingale asset pricing solution given in (16) can be explicitly written as a PDE. With proper initial and boundary conditions, any contingent claim price denominated in the real consumption good is solved by the following PDE

$$\sum_{i,j=1}^{3}\left(\frac{\rho_{ij}v_i v_j}{2}\frac{\partial^2 F}{\partial x_i x_j} + (k_i\theta_i - k_i x_i - \lambda_i)\frac{\partial F}{\partial x_i}\right) - rF = \frac{\partial F}{\partial \tau}, \tag{42}$$

where $\lambda_i$ is the market price of risk of the state variable $x_i$ with respect to the real economy defined as

$$\lambda_i = \left(\frac{J_{ww}}{J_w}\right)\text{Cov}(W, x_i) = \rho_{ia}v_i\sigma_1 + \rho_{in}v_i\sigma_2. \tag{43}$$

The equivalent Martingale measure is to replace the drift $k_i(\theta_i - x_i)$ with the new drift $k_i(\theta_i - x_i - \lambda_i)$. This risk adjustment takes care of the correlation between the state variable and the real wealth growth. Similarly, with initial and boundary conditions, any nominal contingent claim denominated in money will follow

$$\sum_{i,j=1}^{3}\left(\frac{\rho_{ij}\nu_i\nu_j}{2}\frac{\partial^2 G}{\partial x_i x_j} + (k_i\theta_i - k_i x_i - \lambda_i')\frac{\partial G}{\partial x_i}\right) - \iota G = \frac{\partial G}{\partial \tau}, \tag{44}$$

where $\lambda_i'$ is the market price of risk of state variable $x_i$ with respect to the nominal economy defined as

$$\lambda_i' = \left(\frac{-L_{QQ}}{L_Q}\right)\mathrm{Cov}(Q, x_i) = \rho_{ia}\nu_i\sigma_1 + \rho_{in}\nu_i\sigma_2 + \rho_{ip}\nu_i\sigma_3. \tag{45}$$

In this case, the equivalent Martingale is adjusted by $\lambda_i'$, which denotes the correlation between the state variable and the nominal wealth growth. In this economy, the market prices of risk are not affected by the central bank's policy, because all the diffusion coefficients are constants. The policy variable enters the pricing equations through the real and nominal interest rates. We will see these effects more clearly by solving for bond prices and bond options prices in the next subsection.

*Bond and bond options prices*

In this economy, the price of a zero coupon real bond with maturity $\tau$ at time $t$ is the solution to Eq. (42) with terminal condition $F(t+\tau, 0) = 1$. If we let $\varsigma_1 = 1$, $\varsigma_2 = 1$, $\varsigma_3 = q\eta$, then the price of this bond is given by

$$F(t,\tau) = \exp\left(\mathbf{F}(\tau) - \sum_{i=1}^{3}\varsigma_i \mathbf{D}_i(\tau)x_{i,t}\right), \quad \text{where} \tag{46}$$

$$\mathbf{D}_i(\tau) = \frac{1 - e^{-k_i\tau}}{k_i},$$

$$\mathbf{F}(\tau) = c\tau + \sum_{i=1}^{3}\mathbf{G}_i(\tau) + \sum_{i=1}^{2}\sum_{j=i+1}^{3}\mathbf{H}_{ij}(\tau),$$

$$\mathbf{G}_i(\tau) = \left(\frac{\varsigma_i^2\nu_i^2}{2k_i^2} - \varsigma_i\left(\theta_i - \frac{\lambda_i}{k_i}\right)\right)(\tau - D_i(\tau)) - \frac{\varsigma_i^2\nu_i^2}{4k_i}(D_i(\tau))^2,$$

$$\mathbf{H}_{ij}(\tau) = \frac{\rho_{ij}\varsigma_i\varsigma_j\nu_i\nu_j}{k_i k_j}\left(\tau + \frac{1 - e^{(k_i+k_j)\tau}}{k_i + k_j} - D_i(\tau) - D_j(\tau)\right), \quad \text{and}$$

$$c = \rho_{an}\sigma_1\sigma_2 + \sigma_1^2 + \sigma_2^2.$$

The real yield on a T-maturity zero coupon real bond at time $t$ is given by

$$\mathbf{Y}^r(t,\tau) = \sum_{i=1}^{3} \frac{\varsigma_i \mathbf{D}_i(\tau) x_{i,t} - \mathbf{G}_i(\tau)}{\tau} - \sum_{i=1}^{2}\sum_{j=i+1}^{3} \frac{\mathbf{H}_{ij}(\tau)}{\tau} - c. \tag{47}$$

The yield curve generated by this three-factor Vasicek model can show very flexible shapes. The monetary policy variable will affect the yield curve by changing the average yield and the volatility of the yield. It affects the cyclical behavior of the curve through a different linear combination of three factors with unchanged decaying parameters. Unlike most of the term structure models, in this economy, monetary policy has an effect on the real interest rate through the labor market channel. The term premium at time $t$ for maturity $\tau$ is

$$\mathbf{P}^r(t,\tau) = \sum_{i=1}^{3} \frac{\varsigma_i x_{i,t}(\mathbf{D}_i(\tau) - \tau) - \mathbf{G}_i(\tau)}{\tau} - \sum_{i=1}^{2}\sum_{j=i+1}^{3} \frac{\mathbf{H}_{ij}(\tau)}{\tau}. \tag{48}$$

Notice that only the uncertainties related to the state variables matter for the term premia. Unexpected productivity, employment, and price changes do not matter directly except through the correlations with the state variables. In other words, term premia are only affected by uncertainties of the expected changes of economic variables.

In this economy, real bond prices are still affected by inflation. Higher expected inflation generally lowers the prices of real bonds and increases the yields on real bonds when the state variables $x_2$ and $x_3$ are positive. This is because in this case, higher inflation contributes to the real return of production.

A European call option contract $C(t, T, s, K)$ on this time $t$, maturity $s - t$ zero coupon bond with strike price $K$ and expiration date $T$ promises the buyer the right but not the obligation to buy the underlying bond at time $T$ at a predetermined price, $K$ units of the real consumption good. The price of this options contract can be characterized by Eq. (42) with terminal condition

$$C(T, T, s, K) = \max(F(T, s - T) - K, 0). \tag{49}$$

Obviously, we want $s > T$. This bond options contract can be priced as[9]

$$C(t, T, s, K) = F(t, s - t)N(u) - KF(t, T - t)N(u - \sigma_f). \tag{50}$$

where $N$ is the standard Gaussian distribution function and

$$u = \frac{1}{\sigma_f}\log\left(\frac{F(t, s - t)}{F(t, T - t)K}\right) + \frac{\sigma_f}{2},$$
$$\sigma_f^2 = \mathbf{a}'\Sigma\mathbf{a},$$

and where $\mathbf{a}$ is a $3 \times 1$ vector and $\Sigma$ is a $3 \times 3$ matrix given by

$$a_i = \mathbf{D}_i(s - T), \quad \text{and}$$
$$\sigma_{ij} = \rho_{ij}\varsigma_i\varsigma_j\nu_i\nu_j \frac{1 - e^{-(k_i + k_j)(T - t)}}{k_i + k_j}.$$

This options contract in the context of indexed bonds can be understood as if the strike price is also indexed to the CPI so that inflation risk is eliminated from this contract. Thus we can measure the strike price in consumption goods in our model.

Note that both nominal and real interest rates are a constant plus a positive linear combination of three Vasicek factors. Due to the similar structures of the real and nominal interest rates, the pricing formulas for the real and nominal assets are quite alike. We can solve for the price of a nominal zero coupon bond $G(t, \tau)$ with maturity $\tau$ at time $t$ by PDE (44) with terminal condition $G(t + \tau, 0) = 1$ and no boundary condition equivalent to Eq. (46). The solution is

$$G(t, \tau) = \exp\left(\mathbf{F}'(\tau) - \sum_{i=1}^{3} \varsigma_i' \mathbf{D}_i(\tau) x_{i,t}\right), \tag{51}$$

where $\mathbf{F}'$ is similar to that of $\mathbf{F}$ defined in (46) by replacing $\varsigma_i$, $\lambda_i$, and $c$ with $\varsigma_i'$, $\lambda_i'$ and $c'$, where

$$c' = \rho_{an}\sigma_1\sigma_2 + \rho_{ap}\sigma_1\sigma_3 + \rho_{np}\sigma_2\sigma_3 + \sigma_1^2 + \sigma_2^2 + \sigma_3^2,$$

and $\varsigma_1' = 1$, $\varsigma_2' = 1$, and $\varsigma_1' = q$.

The yield on the time-$t$, maturity-$\tau$, zero coupon nominal bond is given by

$$\mathbf{Y}^n(t, \tau) = \sum_{i=1}^{3} \frac{\varsigma_i \mathbf{D}_i(\tau) x_{i,t} - \mathbf{G}'_i(\tau)}{\tau} - \sum_{i=1}^{2} \sum_{j=i+1}^{3} \frac{\mathbf{H}'_{ij}(\tau)}{\tau} - c', \tag{52}$$

where $\mathbf{G}_i'$ and $\mathbf{H}_{ij}'$ are similar to $\mathbf{G}_i$ and $\mathbf{H}_{ij}$ defined in Eq. (46) by replacing $\varsigma_i$ and $\lambda_i$ with $\varsigma_i'$ and $\lambda_i'$. This yield curve is similar to the real yield curve given by (47) except that one of the three factors has a higher unconditional mean and volatility and a different constant. The nominal term premia can be similarly solved as in Eq. (48) by replacing corresponding variables. Again, only state variables matter for the term premia. The prices of options contracts on the nominal bonds are very much like those of the real bonds and need no special exposition. We therefore proceed to discuss this economy in general.

The effect of monetary policy on the prices of nominal bonds is greater than that on the prices of real bonds. When $x_2$ and $x_3$ are positive, higher expected inflation lowers the prices of nominal bonds and increases the yields on them. Compared to nominal bonds, real bond prices are better hedged against inflation. However, nothing is perfectly hedged against inflation in this economy, because inflation has a real effect on real production.

### *Closing the Vasicek model*

In this multi-factor Vasicek model, the idea of how real and nominal economies interplay is explored in a setup with considerable mathematical flexibility. In Vasicek models, because the conditional distributions of the state variables are Gaussian, we can allow all the uncertainties to have all the possible correlation structures. The correlations among unexpected rates of changes of economic variables enter into the interest rates. They are the market risks that are discounted in the interest rates. The correlations between unexpected changes of the state variables and economic variables enter into the market prices of risk. They are required to discount interest rates uncertainties. The correlations among state variables enter into the asset pricing equations. They are to take care of the process by which the interest rates risks are transmitted into asset prices.

Also in this model, the real interest rate reflects only real underlying economic shocks to productivity and natural employment and the real effects of the nominal monetary shocks combined. Nominal interest rates, on the other hand, reflect all the real economic and nominal monetary shocks. We explore this idea with a fixed expected proportion of monetary policies. In spite of the tractability of this model, there are other problems. One of the most obvious drawbacks is that the nominal interest rate can possibly be negative, which presents a serious counterintuitive problem for most of the nominal and real term structure models. One difference this problem can make is that a zero-coupon bond can be traded arbitrarily above its face value depending on some realizations of the state variables. In next section, we will attempt to fix this problem in a different setup. In order to argue that this can happen in our economy, cash is not allowed to be held as an asset. Thus, our nominal economy is acting as a monetary economy without money. In the next section, we fix this problem and allow individuals to hold cash as an asset.

**A nonnegativity constrained model**

In the last section, we assumed that monetary policy affects the economy through one stochastic process by distributing the whole effect proportionally into prices and employment. Here, we propose another model that allows a random distribution effect of monetary policy. Higher inflation rate no longer always raises employment growth. Here we admit the possibility that there might be high inflation and low employment growth. There also can be low inflation together with high employment growth. By choosing square root processes together with OU processes, this model will satisfy the constraints that the nominal interest rate must be non-negative, where real interest rate can be negative and deflation could possibly exist. One advantage of this result is that we implicitly allow individuals to hold cash as an asset. However, because the return of holding cash is always lower than the non-negative return of holding nominal bonds, nobody would choose to hold cash.

*The economy*

In this model, we no longer require the unexpected rates of the changes of productivity, employment, and price to be constants. Their dynamics are

$$dA_t = x_{1,t} A_t\, dt + \sigma_1 \sqrt{x_{1,t}} A_t\, dB_{a,t}, \tag{53}$$

$$dN_t = (x_{2,t} + e^q u_t) N_t dt + \sigma_2 \sqrt{x_{2,t}} N_t dB_{n,t}, \quad \text{and} \tag{54}$$

$$dP_t = e^q (x_{3,t} - u_t) P_t\, dt + \sigma_3 \sqrt{e^q x_{3,t}} P_t\, dB_{p,t}, \tag{55}$$

where $\sigma_i s$ are constants and $q$ is the central banks $\Phi s$ policy variable. $x_i s$ and $u$ are state variables given by

$$dx_{1,t} = k_1(\theta_1 - x_{1,t})\,dt + \nu_1\sqrt{x_{1,t}}\,dB_{1,t}, \tag{56}$$
$$dx_{2,t} = k_2(\theta_2 - x_{2,t})\,dt + \nu_2\sqrt{x_{2,t}}\,dB_{2,t}, \tag{57}$$
$$dx_{3,t} = k_3(\theta_3 - x_{3,t})\,dt + \nu_3\sqrt{x_{3,t}}\,dB_{3,t}, \quad \text{and} \tag{58}$$
$$du_t = k_u(\theta_u - u_t)\,dt + \nu_u\,dB_{u,t}, \tag{59}$$

where the $ks, \theta s$, and $\upsilon s$ are constants. $B_{\cdot,t}s$ are standard Gaussian processes. $B_{a,t}$ and $B_{1,t}$, $B_{n,t}$ and $B_{2,t}$, $B_{p,t}$ and $B_{3,t}$ are correlated with correlation coefficients $\rho_{a1}$, $\rho_{n2}$, and $\rho_{p3}$ respectively. All the other pairs of $B$s are independent of each other. $x_i$ is a square root process. It is mean reverting with unconditional mean $\theta_i$ and decaying parameter $k_i$. The diffusion coefficient is proportional to the square root of its own level. The conditional distribution of this process is non-central $\chi^2$. It can never be negative. This setup also allows the unexpected rates of changes of economic variables to be stochastic and the volatilities are equal to constants times the square roots of the state variables. We do require $\sigma_i^2 < 1$ in order to maintain the long-run growth of the economy. We also want $(1 - \sigma_3^2)\theta_3 > \theta_u$, so that unconditional long-run deflation is excluded from the model.

All agents in the economy can observe the state variables at time $t$, but not the realizations of the Wiener processes. The interplay between inflation and employment is governed by another Vasicek process $u_t$. The central bank can choose the magnitude of the monetary policy variable. With a chosen policy variable $q$, $e^q u_t$ part of the policy will affect the employment, while the rest, $e^q(x_{3,t} - u_t)$, will enter into the expected inflation. The total effect on the nominal interest rate is $e^q x_{3,t}$. Notice the monetary policy is divided into a real effect and a nominal effect stochastically by $u_t$. In the short-run, higher inflation can be either good or bad for employment, but in the long-run, the positive effect dominates. Here we use an exponential function of $q$ in order to restrict the monetary authority from imposing a long-run deflationary policy in exchange for short-run employment gain. We also let $u_t$ be independent of $x_{3,t}$ so that we can have mathematical tractability.

In this economy, expected inflation is given by

$$\omega_t^e = e^q(x_{3,t} - u_t). \tag{60}$$

It is equal to the overall monetary policy effect minus the real effect on the output. The central bank can decide the magnitude of the total effect, but can not decide the real effect and inflation separately.

*Monetary policy*

Let us consider the inflation and employment growth objectives of the central bank. In contrast to the last section, both the employment process and the price process have stochastic volatilities. However, the central bank treats them asymmetrically. It hates both expected inflation and any unexpected change in prices. So the variable to denote the inflation is the sum of expected inflation and the variance of unexpected rate of price change,

$$\pi_t = \mu_{p,t} + \sigma_{p,t}^2 = e^q\left(1+\sigma_3^2\right)x_{3,t} - e^q u_t. \tag{61}$$

The central bank desires a faster rate of employment growth above the natural rate and wishes to avoid fluctuation of the growth rates. Thus, we let the bank's employment objective variable be the expected employment growth minus the variance of the unexpected changes of this growth rate,

$$l_t = \mu_{n,t} - \sigma_{n,t}^2 = \left(1-\sigma_2^2\right)x_{2,t} + e^q u_t. \tag{62}$$

We also assume that the central bank has a zero-inflation and above natural employment growth rates objective. That is

$$\pi_t^* = 0, \quad \text{and} \tag{63}$$
$$l_t^* = \left(1-\sigma_2^2\right)x_{2,t} + b. \tag{64}$$

In this case, the central bank's policy at time $t_0$ is given by

**Proposition 3.** *The central bank will choose the policy variable $q^*$ to be*

$$\begin{aligned} q^* &= \log\left(\frac{b\omega_2 \mathbf{A}_u(t_0)}{(\omega_1+\omega_2)\mathbf{B}_u(t_0) + \omega_1\left(1+\sigma_3^2\right)^2 \mathbf{C}_3(t_0)}\right) && \text{if } u_{t_0} > \frac{-k_u\theta_u}{\delta}, \\ q^* &= -\infty && \text{otherwise.} \end{aligned} \tag{65}$$

*where, $\mathbf{A}_i(s)$ and $\mathbf{B}_i(s)$ are already given in the last section, and*

$$\mathbf{C}_i(s) = \frac{\frac{\theta_i v_i^2}{2k_i} + \theta_i^2}{\delta} + \frac{(x_{i,s}-\theta_i)\left(\frac{v_i^2}{k_i} + 2\theta_i\right)}{k_i+\delta} + \frac{\frac{\theta_i v_i^2}{2k_i} - \frac{x_{i,s}v_i^2}{k_i} + (x_{i,s}-\theta_i)^2}{2k_i+\delta}. \tag{66}$$

**Proof:** See Appendix. □

The central bank will either choose a long-run inflationary policy or a price stabilization policy. It cannot, though, pursue a long-run deflationary policy, because its real policy variable $e^{q^*}$ can not be negative. The case when it chooses a stable policy corresponds to the situation where an inflationary policy does no good to the intertemporal discounted gain from raising the employment growth rate, when there is a large negative realization of $u_t$. The intuition is that in the short-run, the negative effect of inflation on employment dominates. We gain this feature from allowing stochastic influence of inflation on employment. In this model, even if the central bank cannot implement long-run deflationary policy, it does not prevent deflation from happening. In fact, both expected inflation and unexpected inflation are allowed to happen in the short run.

*Interest rates and asset prices*

We can solve for real and nominal interest rates as we did in the last section.

**Proposition 4.** *The real spot interest rate in this economy is given by*

$$r_t = (1-\sigma_1^2)x_{1,t} + (1-\sigma_2^2)x_{2,t} + e^q u_t. \tag{67}$$

*The nominal spot interest rate is given by*

$$i_t = (1-\sigma_1^2)x_{1,t} + (1-\sigma_2^2)x_{2,t} + e^q(1-\sigma_3^2)x_{3,t}. \tag{68}$$

**Proof:** Applying Eqs. (15) and (23) to (53) through (59), the result holds. □

Interest rates reflect not only the expected changes of the underlying economic variables, but also the risk adjusted expected growth rates. In this economy, the real and nominal interest rates are both affected by the productivity and natural employment growth rates. The real interest rates only show the real effects of monetary policy that are expected to be transmitted into employment growth. The nominal interest rates reflect both the real growth of economy and the expected growth of prices. Note that the nominal interest rate is not the sum of real interest rate and expected inflation. This is because the nominal interest rate also takes unexpected inflation part into account. An instantaneous riskless monetary return has to pay a premium for the unexpected price fluctuation.

Another nice feature of our setup is that in this model, the nominal interest rate cannot fall below zero, yet the real interest rate and both expected and actual inflation can be negative. These properties are very basic to these variables, but most people have not captured them in their models. This problem has drawn the attention of Black (1995) who proposed to treat the nominal interest rate as a call option on the sum of real interest rate and expected inflation. In our model, this cannot happen. If the sum of real interest rate and expected inflation minus risk premium of price fluctuation is less than zero, the economy will simply fail, because nobody would devote himself into production because the return of production is lower than the return that could be obtained by simply holding cash. Without production, the price would be driven up until people have incentives to produce again. We argue that in this economy, the real interest rate somehow has to outweigh deflation, or in the case of negative real interest rate, the inflation has to outweigh the real return from production. Otherwise, a market economy will be in disequilibrium.

Let us now derive real and nominal asset prices. With the parametric forms of this model, the PDEs underlying equivalent martingale approach in (16) and (26) can be written out explicitly. With initial and boundary conditions, any asset $H(\cdot)$ that carry returns in real consumption goods has to satisfy

$$\frac{v_u^2}{2}\frac{\partial^2 H}{\partial u^2} + (k_u\theta_u - k_u u)\frac{\partial H}{\partial u} + \sum_{i=1}^{2}\left(\frac{v_i^2 x_i}{2}\frac{\partial^2 H}{\partial x_i^2} + (k_i\theta_i - k_i x_i - \lambda_i' x_i)\frac{\partial H}{\partial x_i}\right) - rH = \frac{\partial H}{\partial \tau}, \tag{69}$$

where $\lambda_i x_i$ is the market price of risk of the state variable $x_i$ with respect to the real economy defined as

$$\lambda_i x_i = \left(\frac{-J_{WW}}{J_W}\right) \mathrm{Cov}\,(W, x_i) = \rho_{ij} v_i \sigma_j x_i, \tag{70}$$

where $j = a$ when $i = 1$ and $j = n$ when $i = 2$. In this model, the market price of risk changes with the state variables. The equivalent martingale measure is achieved by replacing the drift, $k_i(\theta_i - x_i)$, of state the variable $x_i$ with $(k_i + \lambda_i)(k_i\theta_i/(k_i + \lambda_i) - x_i)$. Unlike the Vasicek model, both the unconditional mean and the decaying parameter are adjusted for the state variable. The adjustment takes into account the correlations between the state variables and the real wealth of the consumers.

If asset $I(\cdot)'s$ payoffs are in money, with initial and boundary conditions, the equivalent martingale measure gives us

$$\sum_{i=1}^{3} \left(\frac{v_i^2 x_i}{2}\frac{\partial^2 I}{\partial x_i^2} + (k_i\theta_i - k_i x_i - \lambda_i' x_i)\frac{\partial I}{\partial x_i}\right) - \iota I = \frac{\partial I}{\partial \tau}. \tag{71}$$

Here $\lambda_i' x_i$ is the market price of risk of the state variable $x_i$ with respect to the nominal economy given by

$$\lambda_i' x_1 = \left(\frac{-L_{QQ}}{L_Q}\right)\mathrm{Cov}(Q, x_1) = \rho_{1a} v_1 \sigma_a x_1,$$

$$\lambda_2' x_2 = \left(\frac{-L_{QQ}}{L_Q}\right)\mathrm{Cov}(Q, x_2) = \rho_{2n} v_2 \sigma_n x_2, \quad \text{and}$$

$$\lambda_3' x_3 = \left(\frac{-L_{QQ}}{L_Q}\right)\mathrm{Cov}(Q, x_3) = \sqrt{e^q}\rho_{3p} v_3 \sigma_p x_3.$$

In the nominal economy case, the market price of risk denotes the correlations between the state variables and the nominal wealth of consumers.

Monetary policy affects real asset prices through its effect on the real interest rate. Nominal asset prices are affected in two ways: one through the nominal interest rate; the other through affecting the market price of risk of the third state variable.

*Real and nominal bond prices and bond options prices*

A real zero coupon bond with maturity $\tau$ at time $t$ promises to pay one unit of consumption good at time $t + \tau$. The terminal condition can be written as $H(t + \tau, 0) = 1$. If we let $\varsigma_i = 1 - \sigma_i^2$ for $i = 1, 2$, and $\varsigma_u = e^q$, then the price of this bond is given by

$$H(t, \tau) = \exp(\mathbf{G}_u(\tau) - \varsigma_u \mathbf{D}_u(\tau) u_t) \prod_{i=1}^{2} \mathbf{J}_i(\tau) \exp(-\varsigma_i \mathbf{K}_i(\tau) x_i), \tag{72}$$

where

$$\mathbf{J}_i(\tau) = \left(\frac{2\xi_i e^{(h_i+\xi_i)\tau/2}}{(h_i+\xi_i)(e^{\xi_i\tau}-1)+2\xi_i}\right)^{2k_i\theta_i/v_i^2},$$
$$\mathbf{K}_i(\tau) = \frac{2(e^{\xi_i\tau}-1)}{(h_i+\xi_i)(e^{\xi_i\tau}-1)+2\xi_i},$$
$$h_i = k_i + \lambda_i, \quad \text{and}$$
$$\xi_i = \left(h_i^2 + 2\xi_i v_i^2\right)^{\frac{1}{2}}.$$

The yield of this bond is given by

$$\mathbf{Y}^r = \frac{\varsigma_u \mathbf{D}_u(\tau)u_t - \mathbf{G}_u(\tau)}{\tau} - + \sum_{i=1}^{2} \frac{\varsigma_i \mathbf{K}_i(\tau)x_i - \ln \mathbf{J}_i(\tau)}{\tau}. \tag{73}$$

Again, we have a very flexible curve and the monetary policy variable will affect the mean and variance of the yield. The cyclical behavior of the yield curve movement is also affected. The term premium for maturity $\tau$ bond at time $t$ is given by

$$\mathbf{P}^r = \frac{\varsigma_u u_t(\mathbf{D}_u(\tau)-\tau) - \mathbf{G}_u(\tau)}{\tau} - + \sum_{i=1}^{2} \frac{\varsigma_i x_i(\mathbf{K}_i(\tau)-\tau) - \ln \mathbf{J}_i(\tau)}{\tau}. \tag{74}$$

The real bond prices respond to inflation shocks depending on the sign of the state variable $u_t$. When $u_t$ is positive, real production responds positively to higher inflation. Higher inflation will induce lower real bond prices and higher yields on them. When $u_t$ is negative, real production responds negatively to higher inflation. Higher inflation, in this case, induces higher bond prices and lower yields on them.

Consider the same European call option contract $C(t, T, s, K)$ on the time-$t$, maturity $s - t$, zero-coupon bond with strike price $K$ units of consumption good and expiration date $T$. The strike price is also indexed to the CPI if the return is money. This option contract does not bear any inflation risk. The option pricing problem can be seen as the solution to Eq. (69) with terminal condition

$$C(T, T, s, K) = \max(H(T, s-T) - K, 0). \tag{75}$$

This option contract, at time $t$, is worth[10]

$$C(t, T, s, K) = H(t, s-t)\Phi(d, \omega; \mathbf{m}, \mathbf{n}) - KH(t, T-t)\Phi(d'; \omega'; \mathbf{m}, \mathbf{n}'), \tag{76}$$

where $d$ and $d'$ are scalars and $\omega$, $\omega'$, $m$, $n$ and $n'$ are $2 \times 1$ vectors such that

$$m_i = \frac{4k_i\theta_i}{v_i^2},$$
$$n_i = \frac{2\varsigma_i \varphi_i^2 e^{\xi_i(T-t)} x_{i,t}}{\phi_i + \varphi_i + \mathbf{K}_i(s-T)},$$

$$n_i' = \frac{2\varsigma_i \varphi_i^2 e^{\xi_i(T-t)} x_{i,t}}{\phi_i + \varphi_i},$$
$$\phi_i = \frac{2\xi_i}{\varsigma_i v_i^2 \left(e^{\xi_i(T-t)} - 1\right)},$$
$$\varphi_i = \frac{h_i + \xi_i}{\varsigma_i v_i^2},$$
$$d = \frac{\delta}{\mathbf{D}_u(s-T)\sigma_f},$$
$$d' = \frac{\delta'}{\mathbf{D}_u(s-T)\sigma_f},$$
$$\mu_f = e^{-k_u(T-t)} u_t + \left(1 - e^{-k_u(T-t)}\right)\varsigma_u \theta_u - \frac{\varsigma_u^2 v_u^2 \left(1 - e^{-k_u(T-t)}\right)^2}{2k_u^2},$$
$$\sigma_f^2 = \frac{\varsigma_u^2 v_u^2 \left(1 - e^{-2k_u(T-t)}\right)}{2k_u},$$
$$\omega_i = \frac{2\delta(\phi_i + \varphi_i + \mathbf{K}_i(s-T))}{\mathbf{K}_i(s-T)},$$
$$\omega_i' = \frac{2\delta'(\phi_i + \varphi_i)}{\mathbf{K}_i'(s-T)},$$
$$\delta = \mathbf{G}_u(s-T) + \sum_{i=1}^{2} \log(\mathbf{J}_i(s-T)) - \mathbf{D}_u(s-T)\mu_f + (\mathbf{D}_u(s-T))^2\sigma_f^2 - \log(K).$$

The function $\Phi(u, x; \mathbf{s}, \mathbf{t})$ is given by

$$\int_{-\infty}^{u} \int_0^{x_2 - \frac{z_3 x_2}{u}} \int_0^{x_1 - \frac{x_2 x_1}{x_2} - \frac{z_3 x_1}{u}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z_3^2}{2}} \prod_{i=1}^{2} \chi^2(z_i; s_i, t_i)\, dz_1\, dz_2\, dz_3, \tag{77}$$

where $\chi^2(x; s, v)$ is the distribution function of a noncentral $\chi^2$ distribution with $s$ degree of freedom and noncentrality parameter $v$.

The term structure of nominal interest rates in this case is determined by a three-factor square-root model. A time-$t$, maturity $\tau$, zero-coupon nominal band $I(t, \tau)$ promises to pay one unit of money at time $t + \tau$. The terminal condition is $I(t + \tau, 0) = 1$. If $\varsigma_1' = 1 - \sigma_1^2$, $\varsigma_2' = 1 - \sigma_2^2$, and $\varsigma_3 = e^q(1 - \sigma_3^2)$, the price of this bond is

$$I(t, \tau) = \prod_{i=1}^{3} \mathbf{J}_i'(\tau) \exp(-\varsigma_i' \mathbf{K}_i'(\tau) x_i), \tag{78}$$

where $\mathbf{J}_i'$ and $\mathbf{K}_i'$ are as those given in (72) except replacing $\varsigma_i$ with $\varsigma_i'$.

The yield on this bond is given by

$$\mathbf{Y}^n = \sum_{i=1}^{3} \frac{\varsigma_i' \mathbf{K}_i'(\tau) x_i - \ln \mathbf{J}_i'(\tau)}{\tau}. \tag{79}$$

The term premium for this maturity $\tau$ bond is

$$\mathbf{P}^n = \sum_{i=1}^{3} \frac{\varsigma_i x_i(\mathbf{K}'_i(\tau) - \tau) - \ln \mathbf{J}'_i(\tau)}{\tau}. \tag{80}$$

In this case, the monetary policy influences nominal yield and term premium through a square-root process instead of a Vasicek process like the real economy case.

In the case of nominal bonds, their prices go down uniformly when expected inflation is higher, thus their yields go up. Even if in the special case when the realization of $u_t$ is negative, because the positive effect of the increase in money stock always dominates the negative effects of high inflation on real production, yields on nominal bonds still go up. Because we require the long-run mean of $x_3$ to be greater than the long-run mean of $u_t$, real bond prices respond to inflation shocks less than nominal bond prices. However, real bond prices are not completely inflation proof.

A nominal bond option contract with strike price $K'$ denominated in money can be priced by solving (71) with terminal condition

$$C'(T, T, s, K') = \max(I(T, s - T) - K', 0). \tag{81}$$

The pricing formula is

$$C'(t, T, s, K') = I(t, s - t)\Phi'(\omega; \mathbf{m}, \mathbf{n}) - K' I(t, T - t)\Phi'(\omega'; \mathbf{m}, \mathbf{n}'), \tag{82}$$

where $\omega$, $\omega'$, $m$, $n$ and $n'$ are $3 \times 1$ vectors. The definitions of these terms are similar to those in Eq. (75) by replacing $\varsigma_i$ by $\varsigma'_i$ and

$$\delta = \delta' = \sum_{i=1}^{2} \log(\mathbf{J}'_i(s - T)) - \log(K'), \tag{83}$$

and $\Phi'(x; s, t)$ is given by

$$\int_0^{x_3} \int_0^{x_2 - \frac{z_3 x_2}{x_3}} \int_0^{x_1 - \frac{z_2 x_1}{x_2} - \frac{z_3 x_1}{x_3}} \prod_{i=1}^{3} \chi^2(z_i; s_i, t_i)\, dz_1\, dz_2\, dz_3. \tag{84}$$

Notice that the strike price of the nominal bond option contract has to satisfy

$$K' < \prod_{i=1}^{3} \mathbf{J}'_i(s - T). \tag{85}$$

Otherwise, the strike price would be higher than the maximum possible value of the time-$T$, maturity-$\tau$ bond futures, and the options will be worth zero and will never be exercised. This is one of the consequences of the negativity constraint on the nominal interest rate. However, we do not have an upper limit for the strike prices of the option contracts for the real bond options in this section, and for the strike prices of both the real and nominal bond options in the last section. This is because the real interest rate here and both the real and nominal interest rates in the Vasicek model can be negative. Thus there are no lower bounds for the underlying bond prices.

In this model, we constrain the nominal interest rate not to be negative and at the same time allow the real interest rate and inflation to be negative. This is achieved by observing that inflation must outweigh negative real return and real return must outweigh deflation.

**An empirical description**

In this section, we discuss how the models given in this paper might explain existing empirical findings and how we can possibly use these models to test different conjectures, and to fit real and nominal yield curves.

There have been several empirical studies of term structure models based on modern theories. Those who tested Vasicek and CIR types of models, found that one factor model performed poorly. Some found that two or three factors can describe the yield curve well. In a three-factor model, it is generally found that two of the three factors show very slow mean reversion, and the third factor displays a very strong mean reversion. The first two factors account for the level and slope of the yield curve. The third one is often responsible for variation of short-term interest rates. Brown and Schaefer (1994) worked with U.K. indexed-linked bond data and found that the long-term zero-coupon yield is quite stable.

This paper offers a model that is ready to be tested with nominal bond data, indexed bond data, and inflation data. One can apply either maximum likelihood or different versions of method of moments to nominal bond and indexed bond data together. There are different implications of the model that we can work with. In the Vasicek model case, we can first focus on the test of $\eta$. A zero $\eta$ implies monetary policy is neutral. A non-zero result of $\eta$ will tell us about the real effect of inflation on real production. We believe that the two slow mean reverting processes will correspond to the productivity growth process and the natural employment growth process, $x_{1,t}$, and $x_{2,t}$, and the price growth process third factor will show very fast mean reversion. If we find $\eta$ to have a small magnitude, our model would be consistent with the finding of Brown and Schaefer (1994) that the long-term real yield is much more stable than the long-term nominal yield. Using the Vasicek setup, we can also allow the anticipated and unanticipated uncertainties to be correlated, thus allowing a great flexibility of the estimation and test of correlation coefficients between different variables. We can also use macro survey data, such as inflation forecast data, productivity measurement data, labor market data, and inflation data, because our factors are actually economic variables, unlike most of the term structure literature, which leaves the factors as unspecified state variables.

We can also estimate and test the parametric form of our model offered in section four. Because this model satisfies more realistic conditions, we lose certain generalities like the correlation between different square-root state variables. In actual estimation, we have fewer parameters to estimate. Note that real and nominal interest rates are governed by two common state variables, $x_{1,t}$ and $x_{2,t}$, and one distinctive process, $x_{3,t}$ for nominal interest rates and $u_t$ for real interest rates. The estimation of $x_{3,t}$ and $u_t$ will tell us how monetary policy affects real interest rates on average. It can also tell us how variable the effect of monetary policy on real interest rates can possibly be. Our conjecture about mean reversion properties of the factors is similar to that of the Vasicek model. We suspect that the variance of $u_t$ and $nu_u$, is smaller than the volatility coefficient of $x_{3,t}$ and $v_3$ respectively. The advantage of this model over the Vasicek model is that certain empirical implications of this model are more consistent with some existing empirical findings. For example, Pennacchi (1992) tested a Vasicek type term structure model with inflation forecast survey data and found that expected inflation is positively correlated with real interest rates. In our model, the covariance of current expected inflation given by Eq. (60) and real spot interest rate given by Eq. (67) is

$$\mathrm{Cov}\left(\omega_t^e, r_t\right) = -e^{2q}\,\mathrm{Var}\,(u_t). \tag{86}$$

There are also other results that we can test against the specifications of our model. Similar to the Vasicek case, we can apply survey data in the empirical study of this model as well.

The most important contribution in terms of empirical implication is that it allows us to apply different data sources together. These models also can be used to test different macroeconomics conjectures under the general framework of our model. For example, money non-neutrality would be a valid candidate to be tested. With the indexed bond trading data from U.K., and the future U.S. indexed bond trading data, an estimatable model that allows us to utilize this additional valuable source of information is also very important in terms of finding the volatility structure of interest rates. Thus, we can improve our ability to price interest rate sensitive derivative securities.

## Conclusions

This paper has developed a general equilibrium model of the nominal and real term structure of interest rates. Two examples are offered to illustrate the relationship among the real and nominal economy and monetary policy. Inflation indexed bonds are unlikely to be totally inflation free in this economy. This paper explored the dynamics of interest rates and asset prices in a world where monetary policy is not neutral. In our model, the real interest rate, nominal interest rate, and expected inflation are all correlated. Real assets are affected by real shocks and the real effects of nominal shocks, while nominal assets are subject to both real and nominal shocks. In the three-factor versions of our model, real and nominal interest rates are subject to the influence of two common factors: shocks from productivity and from the labor market. The third factor that affects nominal interest rates is expected inflation uncertainty. And the third factor for real interest rates is the real effect from the expected inflation uncertainty. The first two factors are conjectured to have very slow mean reversion and to govern the level and slope of the yield curve. The third factor takes care of short term shifts of the yield curve. One implication of our model is that long-term real interest rates are more stable than the nominal interest rates. Unless money is neutral, such as in the CIR (1985b) economy, indexed bonds will always be subject to inflation uncertainty.

With inflation indexed bonds introduced into the U.S. capital markets, this paper offered a theoretical approach to understanding the term structure underlying the real and nominal economies. It is shown that there is risk premium in nominal interest rate only to the extent that the correlation between unexpected inflation and unexpected return of real output is nonzero. With more than ten years of trading data from countries such as the U.K., we can actually test this model empirically using nominal and indexed bond data. This feature is an advantage over those models that focus only on either indexed bond data or nominal bond data. We can examine the constraints on the real interest rate, nominal interest rate, expected inflation, actual inflation, and their volatilities. This model can also be used to test hypotheses on monetary economics such as the money neutrality test.

Another contribution of this paper is our attempt to endogenize the price process by studying the central bank's behavior. However, this part of the model employs a static game played at one time only. Adapting this model to the dynamic game case would be a very interesting, although very difficult, further research direction. We also constrained the central bank's objective function to include only inflation and employment. However, we can also add other dimensions to the central bank's behavior such as an exchange rate objective.

## Appendix

*Proof of Proposition 1:*

$$\int_{t_0}^{+\infty} e^{-\delta s}(\omega_1 E_t(\pi_s - \pi_s^*)^2 + \omega_2 E_t(l_s - l_s^*)^2)\, ds. \tag{87}$$

The solution for the monetary policy variable will be given by setting the derivative of (87) with respect to $q$ equal to zero.

*Proof of Proposition 2:*

**Proof:** From (87), the solution $q$ is given by

$$\int_{t_0}^{+\infty} e^{-\delta s}\left(\omega_1 \frac{\partial E_t((1-\eta)q x_{3,s})^2}{\partial q} + \omega_2 \frac{\partial E_t(\eta q x_{3,s} - b)^2}{\partial q}\right) ds = 0. \tag{88}$$

Taking derivative and rearranging the equation, we have

$$\begin{aligned} q &= \left\{\int_{t_0}^{+\infty} e^{-\delta s}(\omega_1(1-\eta)^2 + \omega_2\eta^2)E_{t_0}\left(x_{3,s}^2\right) ds\right\}^{-1} \int_{t_0}^{+\infty} e^{-\delta s}\omega_2\eta b E_{t_0}(x_{3,s})\, ds \\ &= \left\{(\omega_1(1-\eta)^2 + \omega_2\eta^2)\int_{t_0}^{+\infty} e^{-\delta s}\left(\frac{v_3^2}{2k_3}(1-e^{-2k_3 s})\right.\right. \\ &\quad \left.\left. + \left((x_{3,t_0} - \theta_3)e^{-k_3 s} + \theta_3\right)^2\right) ds\right\}^{-1} \omega_2\eta b \int_{t_0}^{+\infty} e^{-\delta s}\left((x_{3,t_0} - \theta_3)e^{-k_3 s} + \theta_3\right) ds \end{aligned}$$

$$= \left\{ (\omega_1 (1-\eta)^2 + \omega_2 \eta^2) \left( \frac{\left(x_{3,t_0} - \theta_3\right)^2 - \frac{v_3}{2k_3}}{2k_3 + \delta} + \frac{2\theta_3 \left(x_{3,t_0} - \theta_3\right)}{k_3 + \delta} + \frac{\frac{v^3}{2k_3} + \theta_3^2}{\delta} \right) \right\}^{-1}$$

$$\times \omega_2 \eta b \left( \frac{x_{3,t_0} - \theta_3}{k_3 + \delta} + \frac{\theta_3}{\delta} \right). \tag{89}$$

□

*Proof of Proposition 4:*

**Proof:** From (88), the solution $q$ is given by solving

$$\int_{t_0}^{+\infty} e^{-\delta s} \left( \omega_1 \frac{\partial E_t \left(e^q \left(1 + \sigma_3^2\right) x_{3,s} - e^q u_s\right)^2}{\partial q} + \omega_2 \frac{\partial E_{t_0} (e^q u_t - b)^2}{\partial q} \right) ds = 0. \tag{90}$$

If there is an interior solution, $q$ is

$$e^q = \left\{ \int_{t_0}^{+\infty} e^{-\delta s} \left( \omega_1 E_{t_0} \left( \left(1 + \sigma_3^2\right)^2 x_s - u_s \right)^2 + \omega_2 E_{t_0} \left(u_s^2\right) \right) ds \right\}^{-1}$$

$$\times \omega_2 b \int_{t_0}^{+\infty} e^{-\delta s} E_{t_0}(u_s)\, ds$$

$$= \left\{ \begin{array}{l} (\omega_1 + \omega_2) \left( \frac{\frac{v_u^2}{2k_u} + \theta_u^2}{\delta} + \frac{2\theta_u \left(u_{t_0} - \theta_u\right)}{k_u + \delta} + \frac{\left(u_{t_0} - \theta_u\right)^2 - \frac{v_u^2}{2k_u}}{2k_u + \delta} \right) \\ \quad + \omega_1 \left(1 + \sigma_3^2\right)^2 \left( \frac{\frac{\theta_3 v_3^2}{2k_3} + \theta_3^2}{\delta} + \frac{\left(x_{3,t_0} - \theta_3\right)\left(\frac{v_3^2}{k_3} + 2\theta_3\right)}{k_3 + \delta} \right. \\ \quad \left. + \frac{\frac{\theta_3 v_3^2}{2k_3} - \frac{x_{3,t_0} v_3^2}{k_3} + \left(x_{3,t_0} - \theta_3\right)^2}{2k_3 + \delta} \right) \end{array} \right\}^{-1}$$

$$\times \omega_2 b \left( \frac{u_{t_0} - \theta_u}{k_u + \delta} + \frac{\theta_u}{\delta} \right),$$

this is when $u_{t_0} > (-k_u \theta_u)/\delta$. Otherwise, the solution is given by setting $q$ to be the boundary value

$$e^q = 0,$$

or $q = -\infty$.

□

*PDE solution forms of real and nominal asset prices*

In section two, we have given equivalent martingale approach to price real and nominal asset prices. We offer the PDEs that are equivalent to the risk-neutral approach. Assume $\mathbf{x}$ is a $k \times 1$ vector of state variables and any state variable $x_i$ follows

$$d\mathbf{x}_i = \mu_i dt + \mathbf{s}_\mathbf{i}' d\mathbf{w}, \tag{92}$$

where $\mathbf{w}$ is an $l \times 1$ vector of standard Wiener processes and $\mathbf{s_i}$ is an $l \times 1$ vector of diffusion coefficient. CIR (1985a) showed that the PDE follow by any real asset price $F$ is given by

$$\begin{aligned}\frac{1}{2}F_{WW} &+ \sum_{i=1}^{k}\text{Cov}(W, x_i)F_{Wx_i} + \frac{1}{2}\sum_{i,j=1}^{k}\text{Cov}(x_i, x_j)F_{x_i x_j} + (rW - C)F_W \\ &+ \sum_{i=1}^{k}\left(\mu_i - \frac{-J_{WW}}{J_W}\text{Cov}(W, x_i) - \sum_{j=1}^{k}\frac{-J_{Wx_j}}{J_W}\text{Cov}(x_i, x_j)\right)F_{x_i} \\ &+ F_t - rF + \delta = 0. \end{aligned} \tag{93}$$

Similarly, in the nominal economy, any asset price $G$ can be solved from PDE

$$\begin{aligned}\frac{1}{2}G_{ZZ} &+ \sum_{i=1}^{k}\text{Cov}(Z, x_i)G_{Zx_i} + \frac{1}{2}\sum_{i,j=1}^{k}\text{Cov}(x_i, x_j)G_{x_i x_j} + (rZ - C)G_Z \\ &+ \sum_{i=1}^{k}\left(\mu_i - \frac{-L_{ZZ}}{L_Z}\text{Cov}(Z, x_i) - \sum_{j=1}^{k}\frac{-L_{Zx_j}}{L_Z}\text{Cov}(x_i, x_j)\right)G_{x_i} \\ &+ G_t - \iota G + \delta = 0. \end{aligned} \tag{94}$$

## Notes

1. See Tobin (1972).
2. We can also let the utility function have only a finite horizon. The result will not change from what we will get in this infinite horizon case.
3. The production function can also be defined by $Y_t = A_t^{\alpha_1} N_t^{\alpha_2}$ where $\alpha_1$ and $\alpha_2$ are relative shares of productivity and labor. Under this definition, our result will not change much except some parameter differences.
4. See Kydland and Prescott (1977), and Rogoff (1985). However, instead of dealing with central bank's credibility problem and its implication for expectation formation, we are dealing with a real tradeoff between output and inflation.
5. We can understand this bond as an indexed bond later after we introduce nominal assets. The payoff of this bond is not directly subject to price fluctuations. Another way to say it is that the principal of this bond is indexed to actual inflation, if we think of the return in monetary terms.
6. The general PDEs are given in Appendix. We will solve the PDEs directly when we solve bond prices later. However, when we solve for bond option prices and other derivative security prices, we use the equivalent martingale method.
7. This approach is first proposed in Harrison and Kreps (1979), and Harrison and Pliska (1981). Constantinides (1992) further developed this approach in his term structure paper. He showed that the state price deflater is the marginal rate of utility.

8. The PDE that a nominal contingent claim has to follow is also given in Appendix.
9. This pricing formula is an extension of the single factor Vasicek case to a correlated three-factor case using equivalent martingale technique. We can also solve this problem under the Heath, Jarrow, and Morton (1990a, 1990b, 1992) framework by starting from the forward rates.
10. The option pricing formulas for a single-factor Vasicek model are given by Jamshidian (1989), and for one- and two-factor CIR models by CIR (1985b), Longstaff and Schwartz (1992), and Chen and Scott (1992). Our solution to the mixed multi-factor model is solved by equivalent Martingale technique.

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